Lebesgue's Theorem on Riemann Integrability

We have seen that any function with finite discontinuity is integrable. Also it is not hard to show that a function with countably many discontinuity is still integrable provided these discontinuous points converges to a single points. On the other hand, functions with too many discontinuous points are not integrable. A typical example is $f(x) = 1$ if x is rational, and $= 0$ otherwise. This function is discontinuous everywhere. In this section we prove Lebesgue fundamental theorem characterizing Riemann integrability in terms of the "size" of discontinuity set.

For any bounded f on $[a, b]$ and $x \in [a, b]$, its **oscillation** at x is defined by

$$\begin{aligned}\omega(f, x) &= \inf_{\delta}\{(\sup f(y) - \inf f(y)) : y \in (x - \delta, x + \delta) \cap [a, b]\} \\ &= \lim_{\delta \to 0^+}\{(\sup f(y) - \inf f(y)) : y \in (x - \delta, x + \delta) \cap [a, b]\}.\end{aligned}$$

It is clear that $\omega(f, x) = 0$ if and only if f is continuous at x. The set of discontinuity of f, D, can be written as

$$D = \bigcup_{k=1}^{\infty} O(k),$$

where $O(k) = \{x \in [a, b] : \omega(f, x) \geq 1/k\}$.

A subset E of $\mathbb{R}$ is of **measure zero** if $\forall \varepsilon > 0, \exists$ a sequence of open intervals $\{I_j\}$ such that

$$E \subseteq \bigcup_{j=1}^{\infty} I_j,$$

and

$$\sum_{j=1}^{\infty} |I_j| < \varepsilon.$$

Proposition 1. *The following statements hold.*
(a) *Any countable set is of measure zero.*
(b) *Any countable union of measure zero sets is again of measure zero.*

Proof. Let $E = \{x_1, x_2, ...\}$ be a countable set. Given $\varepsilon > 0$, the intervals $I_j = (x_j - \frac{\varepsilon}{2^{j+2}}, x_j + \frac{\varepsilon}{2^{j+2}})$ satisfy

$$E \subseteq \bigcup_{j=1}^{\infty} I_j,$$

and

$$\sum_{j=1}^{\infty} |I_j| = \sum_{j=1}^{\infty} \frac{2\varepsilon}{2^{j+2}} = \frac{\varepsilon}{2} < \varepsilon,$$

so E is of measure zero. (a) is proved. (b) can be proved by a similar argument. We leave it as an exercise. $\square$

There are uncountable sets of measure zero. The famous Cantor set is one of them, look up Wikepedia for details. Now, we state the necessary and sufficient condition for Riemann integrability due to Lebesgue.

Theorem 2 (Lebesgue's Theorem)
A bounded function f on $[a, b]$ is Riemann integrable if and only if its discontinuity set is of measure zero.

We shall use the compactness of a closed, bounded interval in the proof of this theorem. Recall that compactness is equivalent to the following property: Let K be a compact set in $\mathbb{R}$. Suppose that $\{I_j\}$ is a sequence of open intervals satisfying $K \subseteq \bigcup_{j=1}^{\infty} I_j$. Then we can choose finitely many intervals $I_{j_1}, ..., I_{j_N}$ so that $K \subseteq I_{j_1} \cup \cdots \cup I_{j_N}$.

Proof. Suppose that f is Riemann integrable on $[a, b]$. Recall the formula

$$D = \bigcup_{k=1}^{\infty} O(k).$$

By Proposition 1 (b) it suffices to show that each $O(k)$ is of measure zero. Given $\varepsilon > 0$, by Integrability Criterion I, we can find a partition P such that

$$\overline{S}(f, P) - \underline{S}(f, P) < \varepsilon/2k.$$

Let J be the index set of those subintervals of P which contains some elements of $O(k)$ in their interiors. Then

$$\begin{aligned}
\frac{1}{k} \sum_{j \in J} |I_j| &\le \sum_{j \in J} (\sup_{I_j} f - \inf_{I_j} f)\Delta x_j \\
&\le \sum_{j=1}^{n} (\sup_{I_j} f - \inf_{I_j} f)\Delta x_j \\
&= \overline{S}(f, P) - \underline{S}(f, P) \\
&< \varepsilon/2k.
\end{aligned}$$

Therefore
$$\sum_{j\in J} |I_j| < \varepsilon/2.$$
Now, the only possibility that an element of $O(k)$ is not contained by one of these I_j is it being a partition point. Since there are finitely many partition points, say N, we can find some open intervals $I'_1, ..., I'_N$ containing these partition points which satisfy
$$\sum |I'_i| < \varepsilon/2.$$
So $\{I_j\}$ and $\{I'_i\}$ together form a covering of $O(k)$ and its total length is strictly less than ε. We conclude that $O(k)$ is of measure zero.

Conversely, given $\varepsilon > 0$, fix a large k such that $\frac{1}{k} < \varepsilon$. Now the set $O(k)$ is of measure zero, we can find a sequence of open intervals $\{I_j\}$ satisfying
$$O(k) \subseteq \bigcup_{j=1}^{\infty} I_j,$$
$$\sum_{j=1}^{\infty} |I_{i_j}| < \varepsilon.$$
One can show that $O(k)$ is closed and bounded, hence it is compact. As a result, we can find $I_{i_1}, ..., I_{i_N}$ from $\{I_j\}$ so that
$$O(k) \subseteq I_{i_1} \cup ... \cup I_{i_N},$$
$$\sum_{j=1}^{N} |I_j| < \varepsilon.$$
Without loss of generality we may assume that these open intervals are mutually disjoint since, whenever two intervals have nonempty intersection, we can put them together to form a larger open interval. Observe that $[a, b] \setminus (I_{i_1} \cup \cdots \cup I_{i_N})$ is a finite disjoint union of closed bounded intervals, call them $V'_i s,\ i \in A$. We will show that for each $i \in A$, one can find a partition on each $V_i = [v_{i-1}, v_i]$ such that the oscillation of f on each subinterval in this partition is less than $1/k$.

Fix $i \in A$. For each $x \in V_i$, we have
$$\omega(f, x) < \frac{1}{k}\,.$$
By the definition of $\omega(f, x)$, one can find some $\delta_x > 0$ such that
$$\sup\{f(y) : y \in B(x, \delta_x) \cap [a, b]\} - \inf\{f(z) : z \in B(x, \delta_x) \cap [a, b]\} < \frac{1}{k}\,,$$

where $B(y, \beta) = (y - \beta, y + \beta)$. Note that $V_i \subseteq \bigcup_{x \in V_i} B(x, \delta_x)$. Since V_i is closed and bounded, it is compact. Hence, there exist $x_{l_1}, \ldots, x_{l_M} \in V_i$ such that $V_i \subseteq \bigcup_{j=1}^{M} B(x_{i_j}, \delta_{x_{l_j}})$. By replacing the left end point of $B(x_{i_j}, \delta_{x_{l_j}})$ with v_{i-1} if $x_{l_j} - \delta_{x_{l_j}} < v_{i-1}$, and replacing the right end point of $B(x_{i_j}, \delta_{x_{l_j}})$ with v_i if $x_{l_j} + \delta_{x_{l_j}} > v_i$, one can list out the endpoints of $\{B(x_{l_j}, \delta_{l_j})\}_{j=1}^{M}$ and use them to form a partition S_i of V_i. It can be easily seen that each subinterval in S_i is covered by some $B(x_{l_j}, \delta_{x_{l_j}})$, which implies that the oscillation of f in each subinterval is less than $1/k$. So, S_i is the partition that we want.

The partitions S_i's and the endpoints of $I_{i_1}, ..., I_{i_N}$ form a partition P of $[a, b]$. We have

$$\begin{aligned}
\overline{S}(f, P) - \underline{S}(f, P) &= \sum_{I_{i_j}} (M_j - m_j)\Delta x_j + \sum (M_j - m_j)\Delta x_j \\
&\leq 2M \sum_{j=1}^{N} |I_{i_j}| + \frac{1}{k} \sum \Delta x_j \\
&\leq 2M\varepsilon + \varepsilon(b - a) \\
&= [2M + (b - a)]\varepsilon,
\end{aligned}$$

where $M = \sup_{[a,b]} |f|$ and the second summation is over all subintervals in $V_i, i \in A$. By Integrability Criterion I, f is integrable on $[a, b]$. $\square$